Dual Discrete Geometry Methods Based on Weighted Triangulations for Capacitance Extraction

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The accuracy and convergence of the discrete geometric method (DGM) in terms of scalar potential depend on the quality of spatial meshes. To improve performance of the DGM based on existing meshes, the concept of weighted triangulations is reviewed and introduced to optimally transport the circumcenter of element to a well-centered position which results in the modified formulas of Hodge operation. The dual DGMs based on weighted triangulations are reformulated over dual meshes. They both help to reduce the spatial discretization errors and accelerate electromagnetic field analysis to extract circuit parameters. The electrostatic field and capacitance extraction examples are studied. The numerical results demonstrate that the accuracy and stability of the DGM are improved.

***Index Terms*—Discrete Hodge operator, discrete geometric method, parasitic capacitance extraction, weighted triangulations.**

I. Introduction

RECENTLY, the discrete geometric method (DGM) has attracted much attention in electromagnetic field analysis and in the applications such as the capacitance extraction in integrated circuits (IC) [1]. For such type of method, the accuracy, convergence and stability depend on the shape and size of the space discretization, so that the mesh generation algorithm is critical [2]. However, in the case of complicate IC structures, there is yet unlikely a versatile algorithm to generate perfect meshes. Improving mesh quality requires much effort and time cost.

The DGM has advantages due to its simple implementation and the energy complementarity when the dual DGM established on the dual meshes. In addition, both the dual formulations work with the scalar potential [1]. The interlocked dual meshes are usually based on barycenter or circumcenter. The circumcentric duals are preferred for the orthogonality in nature. However, in the case of bad shaped elements, the circumcenter may dropped out of element and make the matrix ill-conditioned. On the other hand, the barycentric duals guarantee that the vertices of the dual mesh lies inside the corresponding primal elements. But the fact of losing the orthogonality brings the discretization errors. To alleviate these issues, a family of well-shaped primal-dual pairs of meshes, *i.e.*, weighted triangulations, or weighted dual, is introduced for fast and accurate computations in [3]. In this paper, this method is reviewed, implemented, and applied for capacitance extraction of IC interconnects. Its energy complementary and duality are studied.

II. Discrete Geometric Method

A. Dual Formulations on Dual Meshes

Let $\mathcal{M}_P$ and $\mathcal{M}_D$ be the two interrelated primal and dual meshes, as shown in Fig. 1, where $\mathcal{M}_P$ is usually a triangular mesh and $\mathcal{M}_D$ is formed through Hodge mapping. In the framework of discrete geometry, the field equations are written algebraically on them. For an electrostatic problem, $\operatorname{curl}\mathbf{e} = 0$ and $\operatorname{div}\mathbf{d} = \rho$. Attaching $\mathbf{e}$ to the edges of $\mathcal{M}_P$ and $\mathbf{d}$ to the edges (in two dimension) or the facets (in three dimension) of $\mathcal{M}_D$, or *vice versa*, and using the Hodge discrete version of the constitutive matrices M_ε^P and M_ε^D, we obtain the primal and dual sets of equations as [1]

$$\mathrm{G}^T M_\varepsilon^P \mathrm{G}\bar{v}^P = \bar{\rho}^D \text{, and } \mathrm{D}M_\varepsilon^D \mathrm{D}^T \bar{v}^D = \bar{\rho}^P,$$

where G and D are incident matrices, and the scalar potential $\bar{v}^P$ and $\bar{v}^D$ are nodal degrees of freedom on $\mathcal{M}_P$ and $\mathcal{M}_D$, respectively.

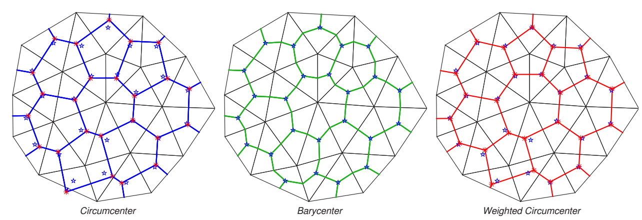


Fig. 1. Primal meshes $\mathcal{M}_P$ with triangle elements and dual meshes $\mathcal{M}_D$ in two dimension. $\mathcal{M}_P$-s are colored with black. $\mathcal{M}_D$-s of (a) in blue, (b) in green, and (c) in red are built using circumcenter, barycenter and weighted circumcenter, respectively. The barycenter is marked with pentacles in blue.

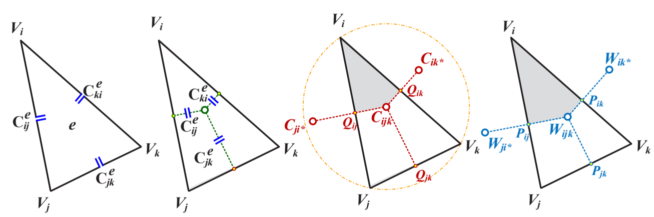


Fig. 2. Equivalent elementary capacitance contributions along three edges on the primal mesh (a) and the dual mesh (b); the circumcenter C_{ijk} (c) and the weighted circumcenter W_{ijk} (d) are used to evaluate dual relations.

B. Weighted Triangulations

In the concept of weighted triangulations, the position of the original circumcenter (Fig. 1a) is shifted to a weighted circumcenter (Fig. 1c). The latter is closer to barycenter but the dual meshes are kept mutually orthogonal. Then the geometric dimensions used for calculating constitutive matrices needs to be modified.

The primal elementary capacitance on edge ij based on the primal formulation using circumcenter within primal mesh, as shown in Fig. 2a and Fig. 2c is given by

$$C_{ij}^{e} = \varepsilon\,\overline{\mathrm{Q}_{ij}\mathrm{C}_{ijk}}/\,\overline{\mathrm{V}_i\mathrm{V}_j}\,, \tag{1}$$

within weighted triangulations in Fig. 2d it is replaced by

$$C_{ij}^{e} = \varepsilon\,\overline{\mathrm{W}_{ijk}\mathrm{P}_{ij}}/\,\overline{\mathrm{V}_i\mathrm{V}_j}\,, \tag{2}$$

where by denoting w_s as the weight value on vertex s, and I, J, and K as inner angles located at vertex i, j, and k respectively, the dimension of edge on primal mesh and on dual mesh are given respectively as

$$\overline{\mathrm{V}_i\mathrm{P}_{ij}} = \frac{\overline{\mathrm{V}_i\mathrm{V}_j}}{2} + \frac{w_i - w_j}{2\overline{\mathrm{V}_i\mathrm{V}_j}}, \quad \overline{\mathrm{P}_{ij}\mathrm{V}_j} = \frac{\overline{\mathrm{V}_i\mathrm{V}_j}}{2} + \frac{w_j - w_i}{2\overline{\mathrm{V}_i\mathrm{V}_j}}, \text{and}$$

$$\overline{\mathrm{W}_{ijk}\mathrm{P}_{ij}} = \frac{\overline{\mathrm{V}_i\mathrm{V}_j}}{2}\cot\mathrm{K} + \frac{(w_j - w_k)\cot\mathrm{I} + (w_i - w_k)\cot\mathrm{J}}{2\overline{\mathrm{V}_i\mathrm{V}_j}}\,.$$

To perform weighted triangulation without changing any vertex position, the optimal weights imposed on all vertexes that guarantee the minimization of errors of global energy are given by

$$w_i^{*} = \frac{\sum_{\mathrm{t}_{ijk}\in\Omega(i)}(w_j\cot\mathrm{K} + w_k\cot\mathrm{J} + 2\mathrm{t}_{ijk}(\mathbf{c}_{\mathrm{t}} - \mathbf{b}_{\mathrm{t}})\cdot\mathbf{n}_i)}{\sum_{\mathrm{t}_{ijk}\in\Omega(i)}\frac{\left(\overline{\mathrm{V}_i\mathrm{V}_j}\right)^2}{2\mathrm{t}_{ijk}}}, \tag{3}$$

where $\mathbf{n}_i$ is the unitary exterior normal vector on edge jk opposite vertex i. $\mathbf{c}_{\mathrm{t}}$ and $\mathbf{b}_{\mathrm{t}}$ are, respectively, the circumcenter and the barycenter of the triangle element ijk, whose area is t_{ijk} [3][4].

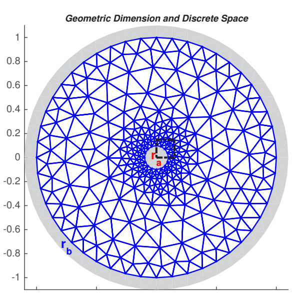
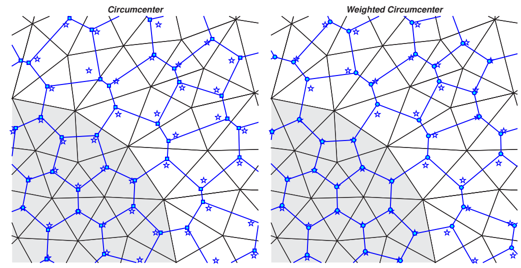


Fig. 3. Geometric model of a cylindrical capacitor example and discrete mesh space, where the relative permittivity of the dielectric between the plates is $\varepsilon_r = 3.70$. 0-volt and 1-volt potentials are imposed on the two plates; The unit of dimension is μm.

III. Example and Discussion

To validate the dual DGMs using weighted triangulations and to investigate their energy complementary characteristics, an electrostatic problem of a cylindrical capacitor as shown in Fig. 3, for which the analytical solution is available, is studied. The computation domain is meshed into triangle elements, and then the weighted circumcenters are determined using Eq. 3 to build the dual mesh. It can be noticed that the weighted circumcenters in Fig. 4 are closer to the barycenters.



(*a*) (*b*)

Fig. 4. Dual meshes generated through triangulation (a) and weighted triangulation (b). The region shown here corresponds to the region with bold frame in Fig. 3.

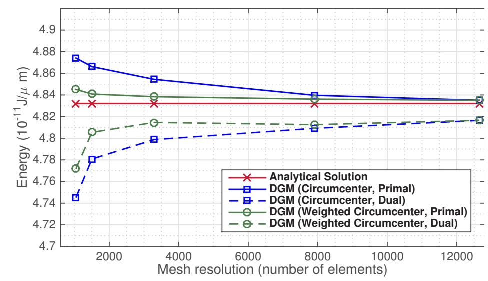


Fig. 5. Energy bounds for dual DGMs using circumcenter and weighted circumcenter, and energy compared with analytic solution.

The numerical solutions given by dual DGMs over traditional meshes and weighted meshes are compared with the analytical solution in Fig. 5.

According to Fig. 3, both the energy bounds and the consistency, *i.e.*, the convergence along with the mesh refinement, are clearly observed. For the primal DGM and the dual DGM, the weight triangulation achieves better accuracy, because it reduces the space discretization error. It should be also noticed that the average curves of dual DGMs on both original meshes and weighted meshes almost coincide. This shows that the dual DGMs can collectively reduce the space discretization error.

IV. Conclusion

The dual DGMs work on the meshes processed by weighted triangulations, which are proven to reduce discretization error effectively. The energy complementary is observed. Three-dimensional results will be provided in the full paper. Since the dual methods and the weighted triangulations are both aimed at the reduction of discretization errors, the cost for weighted triangulations should be evaluated and compared with the paired DGMs.

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